Quantum systems

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Wigner Crystal

[E. P. Wigner, Phys. Rev. 46, 1002 (1934)]

$n \sim a^{-d}$ electrons per unit volume.

- Kinetic energy per electron $K \sim \frac{\hbar^2}{2ma^2}$
- Potential energy per electron $U \sim \frac{e^2}{4\pi\epsilon_0 a}$

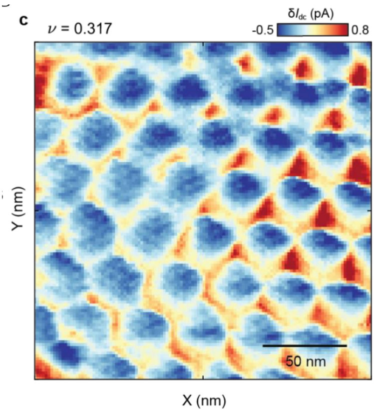
$a \gg \frac{2\pi\epsilon_0\hbar^2}{2m} \Rightarrow$ minimize potential energy
$\Rightarrow$ Formation of a close-packed lattice

How to favor Wigner crystallization ?

$\rightarrow$ reduce kinetic energy

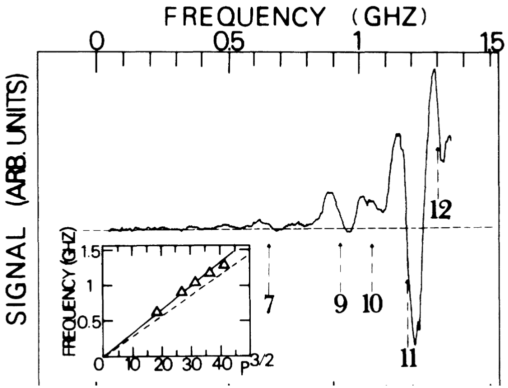
1. Magnetic field [quench kinetic energy in a 2D Landau Level]
2. Flat or quasi-flat bands [magic angle graphene]

Experimental realization in bilayer graphene



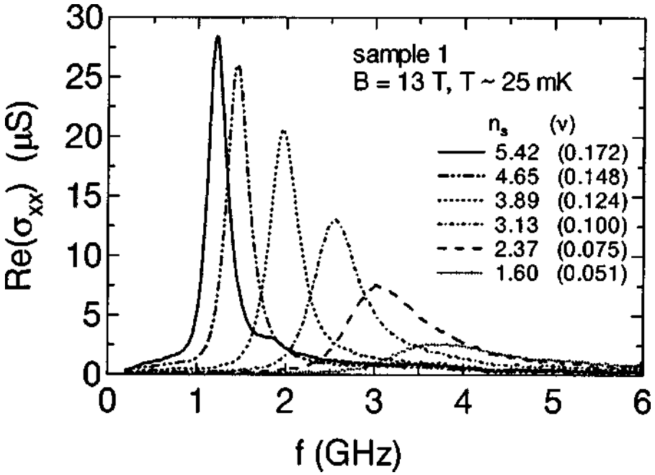
from Tsui et al. Nature **628**, 287 (2024)

Detection of a shear mode in GaAs:GaAlAs heterojunction



from E. Y. Andrei et al. Phys. Rev. Lett. **60**, 2765 (1988)

Presence of a pinning peak in the conductivity



from Li et al. Phys. Rev. B **61**, 10905 (2000)

Elastic energy

Isotropic elasticity, short range interaction

$$
\begin{aligned}
u_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
V_{el} &= \int d^d\vec{r}\,\frac{E}{2(1+\sigma)}\left[\sum_{ij} u_{ij}^2 + \frac{\sigma}{1-2\sigma}\left(\sum_i u_{ii}\right)^2\right],
\end{aligned}
$$

E = Young modulus
σ = Poisson coefficient

Lagrangian

Kinetic energy

$$T = \sum_i \frac{\rho}{2} \int d^d \vec{r} \left(\frac{\partial u_i}{\partial t} \right)^2 ,$$

Lagrangian

$$\begin{aligned} L &= T - V_{el} \\ &= \sum_i \frac{\rho}{2} \int d^d \vec{r} \left(\frac{\partial u_i}{\partial t} \right)^2 \\ &\quad - \int d^d \vec{r} \frac{E}{2(1+\sigma)} \left[\sum_{ij} u_{ij}^2 + \frac{\sigma}{1-2\sigma} \left(\sum_i u_{ii} \right)^2 \right] , \end{aligned}$$

Path integral quantization

Feynman formula

$$\langle \{u_i'\} | e^{-iHt/\hbar} | \{u_i\} \rangle = \int \mathcal{D}u_j e^{\frac{i}{\hbar}\int L dt}$$

Wick rotation/Matsubara formalism

- Analytic continuation $t \to -i\tau$
- $e^{-iHt/\hbar} \to e^{-\tau H/\hbar}$
- $\partial_t u_i \to i\partial_\tau u_i$ in L
- $idt/\hbar \to d\tau/\hbar$

$\Rightarrow \langle \{u_i'\} | e^{-\frac{H}{k_B T}} | \{u_i\} \rangle$ as path integral with $\tau = \frac{\hbar}{k_B T}$.
Partition function by taking $u_i = u_i'$ and integrating over $u_i(\vec{r})$.

Partition function in Matsubara time

Path integral in Matsubara time

$$
\begin{aligned}
Z &= \int_{u_j(\vec{r},0)=u_j(\vec{r},\beta\hbar)} \mathcal{D}u_j e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} L_M d\tau} \\
L_M &= \sum_i \frac{\rho}{2}\int d^d\vec{r}\left(\frac{\partial u_i}{\partial\tau}\right)^2 \\
&\quad + \int d^d\vec{r}\frac{E}{2(1+\sigma)}\left[\sum_{ij} u_{ij}^2 + \frac{\sigma}{1-2\sigma}\left(\sum_i u_{ii}\right)^2\right],
\end{aligned}
$$

$\Rightarrow$ classical partition function of a slab of thickness $\beta\hbar$ in a $(d+1)-$dimensional space.

Pinning potential

Gaussian random potential

$$\overline{V(\vec{r})V(\vec{r}')} = D\delta(\vec{r}-\vec{r}')$$

$$L_{disorder} = -\int d^d\vec{r}\,V(\vec{r})\rho(\vec{r})$$

$$\rho(\vec{r}) = \sum_{\vec{R}} \delta(\vec{r}-\vec{R}+\vec{u}(\vec{R}))$$

Density as a Dirac comb

Rewriting the displacement $\vec{u}$

$$\vec{x} + \vec{u}(\vec{x}) = \vec{y} \quad \Leftrightarrow \quad \vec{y} - \vec{\eta}(\vec{y}) = \vec{x}$$

$$\delta(\vec{x} + \vec{u}(\vec{x}) - \vec{y}) \quad = \quad \delta(\vec{y} - \vec{\eta}(\vec{y}) - \vec{x}) \det\left(\mathrm{Id} - \frac{\partial \vec{\eta}}{\partial \vec{y}}\right)$$

Perturbatively,

$$\eta(\vec{y}) \simeq \vec{u}(\vec{y})$$

$$\det\left(\mathrm{Id} - \frac{\partial \vec{\eta}}{\partial \vec{y}}\right) \simeq 1 + \vec{\nabla}_y \cdot \vec{u}$$

$$\rho(\vec{r}) \simeq (1 + \vec{\nabla}_y \cdot \vec{u}) \sum_{\vec{R}} \delta(\vec{r} - \vec{u}(\vec{r}) - \vec{R})$$

Poisson summation formula

Dirac comb as sum on reciprocal lattice vectors

$$\sum_{\vec{R}} \delta(\vec{R} - \vec{a}) = \frac{1}{(\vec{a} \times \vec{b}) \cdot \vec{c}} \sum_{\vec{G}} e^{i\vec{G}\cdot\vec{a}}$$

$$\vec{G} \cdot \vec{R} \in 2\pi\mathbb{Z}$$

Expression of the density

$$\rho(\vec{r}) \simeq \frac{(1 + \vec{\nabla} \cdot \vec{u})}{(\vec{a} \times \vec{b}) \cdot \vec{c}} \sum_{\vec{G}} e^{i\vec{G}\cdot[\vec{r} - \vec{u}(\vec{r})]}$$

Rewriting the pinning potential

With Poisson summation

$$L_{\mathrm{disorder}} = \int d^d\vec{r}\left[V_0(\vec{r})(\vec{\nabla}\cdot\vec{u}) + \sum_{\vec{G}\neq\vec{0}} V_{\vec{G}}(\vec{r})e^{i\vec{G}\cdot\vec{u}(\vec{r})}\right]$$

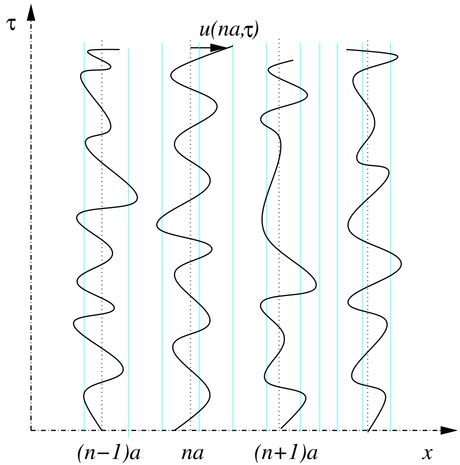
Disorder is (Matsubara) time-independent $\Rightarrow$ classical system in $(d+1)$ dimensions with columnar defects

In one dimension

Matsubara action

$$S = \int_0^{\beta\hbar} d\tau \int dx \left[\frac{\rho}{2} \left(\frac{\partial u}{\partial \tau} \right)^2 + \frac{\kappa}{2} \left(\frac{\partial u}{\partial x} \right)^2 + V_0(x) \left(\frac{\partial u}{\partial x} \right) + \sum_{n \neq 0} V_n(x) e^{i \frac{2\pi n u(x,\tau)}{a}} \right]$$

Equivalence with 2D elasticity and columnar disorder



Relation with disordered Tomonaga-Luttinger liquid

[Giamarchi and Schulz Phys. Rev. B **37**, 325 (1988)]

$$S = \int \frac{dx d\tau}{2\pi K}\left[v(\partial_x\phi)^2 + \frac{(\partial_\tau\phi)^2}{v}\right] + \int dx d\tau V(x)\rho(x)$$

$$\rho(x) = \rho_0 - \frac{\partial_x\phi}{\pi} + \sum_{m\neq 0} A_m e^{2im(\phi(x)-2k_F x)}$$

$$\overline{V(x)V(x')} = D\delta(x-x')$$

Mapping

$$\phi(x,\tau) = \frac{\pi u(x,\tau)}{a}$$

$$v = \sqrt{\frac{\kappa}{\rho}} \qquad K = \frac{\pi\hbar}{a^2\sqrt{\kappa\rho}}$$

Giamarchi-Schulz transition

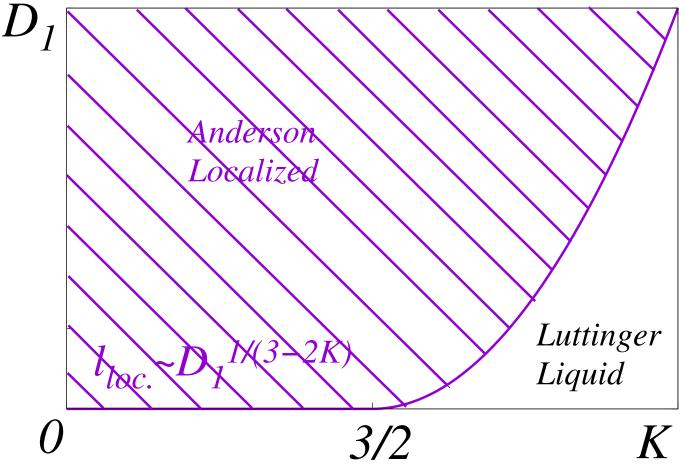
Statistical tilt symmetry

$\phi(x) \to \phi(x) + \int_0^x \frac{KV_0(y)}{\pi v} dy$ leaves $V_{n\neq 0}(x)$ invariant.

Most relevant perturbation for $n = 1$

$$\begin{aligned} \overline{V_n(x)V_n^*(x')} &= D_n \delta(x - x') \\ \frac{dD_n}{dl} &= (3 - 2n^2 K) D_n \\ \frac{dK}{dl} &= -K^2 D_1 + \dots \end{aligned}$$

Zero temperature phase diagram



Anderson localized phase

Replica trick

$$\overline{Z^n} = \int \prod_{a=1}^{n} \mathcal{D}u_a \overline{e^{-\sum_{a=1}^{n} S[u_a]}}$$

$$\overline{F} = -\frac{1}{\beta}\overline{\ln Z} = -\frac{1}{\beta}\lim_{n\to 0}\frac{\overline{Z^n}-1}{n}$$

$$\overline{\langle u_a(x,\tau)u_a(x',0)\rangle} = \lim_{n\to 0} \overline{Z^{n-1}\int \mathcal{D}u_a e^{-S[u_a]} u_a(x,\tau)u_a(x',0)}$$

$$\overline{\langle u_a(x)\rangle\langle u_b(x')\rangle} = \lim_{n\to 0} \overline{Z^{n-2}\int \mathcal{D}[u_a,u_b] e^{-S[u_a]-S[u_b]} u_a(x)u_b(x')}$$

Replicated action

[Giamarchi and Le Doussal Phys. Rev. B **53**, 15206 (1996)]

$$
\begin{aligned}
S &= \sum_{a=1}^{n} \int_0^{\beta\hbar} d\tau \int dx \left[\frac{\rho}{2} \left(\frac{\partial u_a}{\partial \tau} \right)^2 + \frac{\kappa}{2} \left(\frac{\partial u_a}{\partial x} \right)^2 \right] \\
&\quad - D_1 \sum_{a,b} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \int dx \cos \frac{2\pi}{a} \left[u_a(x,\tau) - u_b(x,\tau') \right]
\end{aligned}
$$

Gaussian Variational Ansatz

[M. Mézard G. Parisi J. Phys. I **1**, 809 (1991)]

$$
\begin{aligned}
S_G &= \frac{1}{\beta\hbar}\sum_{\omega_n=\frac{2\pi n}{\beta\hbar}}\sum_{a,b}\int\frac{dq}{2\pi}u_a(q,\omega_n)G^{-1}_{ab}(q,\omega_n)u_b(-q,-\omega_n)\\
F_G &= -\frac{1}{\beta}\ln\left[\int\mathcal{D}[u_a]e^{-S_G}\right]\\
F_{var.} &= F_G+\langle S-S_G\rangle_{S_G}\geq\overline{F}
\end{aligned}
$$

$\Rightarrow$ minimize $F_{var.}$ with respect to G_{ab}

Variational method

Saddle point condition [Giamarchi Le Doussal Phys. Rev. B 53, 15206 (1996)]

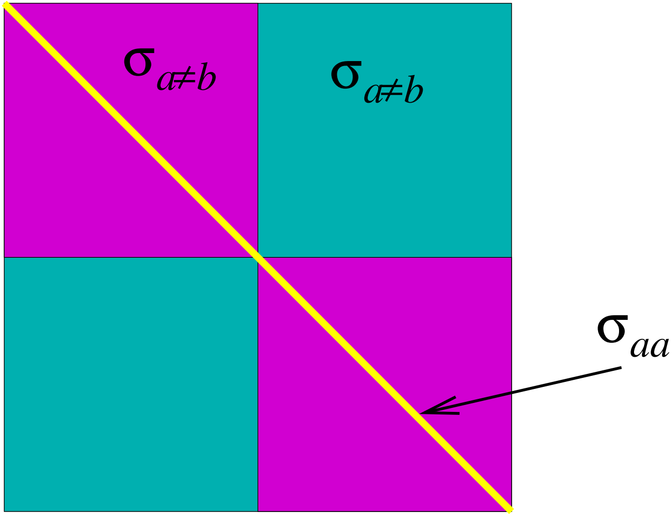
$$(G^{-1})_{ab}(q,\omega_n) = \frac{\omega_n^2+(uq)^2}{\pi u K}\delta_{ab} - \sigma_{ab}(\omega_n)$$

$$\sigma_{a\neq b}(\omega_n) = \frac{4D\beta}{(\pi\alpha\hbar)^2} e^{-2[G_{aa}(0,0)+G_{bb}(0,0)-2G_{ab}(0,0)]}\delta_{\omega_n,0}$$

$$\sigma_{aa}(\omega_n) = \frac{4D}{(\pi\alpha\hbar)^2}\left[2\sum_{b\neq a} e^{-2[G_{aa}(0,0)+G_{bb}(0,0)-2G_{ab}(0,0)]} + \int_0^\beta e^{-4[G_{aa}(0,0)-G_{aa}(0,\tau)]}(1-\cos(\omega_n\tau))\right]$$

need to take $n \to 0$

One-step ansatz for the variational method



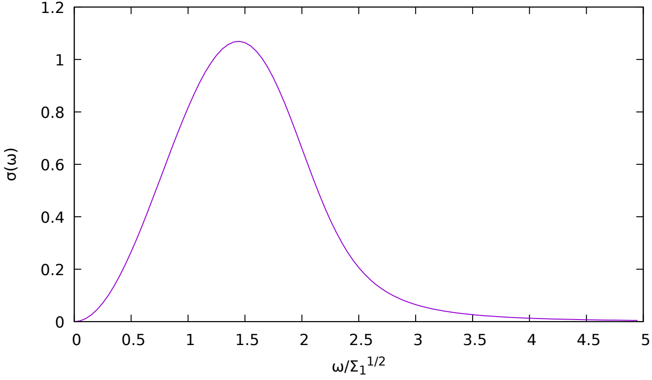
Symmetry between replicas is broken in the localized phase.

Main result

ac conductivity for $K \to 0$

$$
\begin{aligned}
j_a(x,\tau) &= \frac{e\rho}{m}\partial_\tau u_a(x,\tau) \\
\Sigma_1 &\sim D^{2/3} \\
I(\omega) &= \Sigma_1 f(i\omega/\sqrt{\Sigma_1}) \\
f(u) &= 2\left[1 - \frac{1}{\sqrt{1+u^2+f(u)}}\right] \\
\sigma(\omega) &\sim \frac{i\omega}{-\omega^2+\Sigma_1+I(\omega)} \\
\mathrm{Re}\,\sigma(\omega) &\sim \omega^2 \; (\omega \to 0)
\end{aligned}
$$

Real part of conductivity in the 1D case



Two dimensions and above

In two dimensions

- Disordered phase with one-step replica symmetry breaking
- $\Sigma_1 \sim D$
- $\overline{\langle (u_a(x,\tau) - u_a(0,0))^2 \rangle} \sim \ln(D^{1/2}|x|)$ (QLRO as $x \to +\infty$)

In three dimensions

- Full replica symmetry breaking

The two-dimensional Wigner crystal

Coulomb interaction

$$
\begin{aligned}
U &= \frac{e^2}{8\pi\epsilon_0}\int d^2\vec{r} d^2\vec{r}' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\
&= \frac{e^2}{4\epsilon_0}\int \frac{d^2\vec{q}}{(2\pi)^2}\int \frac{d^2\vec{q}'}{(2\pi)^2}\frac{\rho(\vec{q})\rho(-\vec{q})}{|\vec{q}|} \\
&\simeq \frac{e^2\rho_0^2}{4\epsilon_0}\int \frac{d^2\vec{q}}{(2\pi)^2}\int \frac{d^2\vec{q}'}{(2\pi)^2}\frac{[\vec{q}\cdot\vec{u}(\vec{q})][\vec{q}\cdot\vec{u}(-\vec{q})]}{|\vec{q}|}
\end{aligned}
$$

Magnetic field

$$
\begin{aligned}
\vec{A}(\vec{R}+\vec{u}) &= \frac{1}{2}\vec{B}\times(\vec{R}+\vec{u}) \\
\vec{\jmath}\cdot\vec{A} &= \frac{\rho_0 e}{2}\vec{B}\times(\vec{R}+\vec{u})\cdot\partial_t\vec{u} \\
&= \frac{\rho_0 e}{2}\left[\vec{B}\cdot(\vec{u}\times\partial_t\vec{u}) + \partial_t(\vec{B}\cdot(\vec{R}\times\vec{u}))\right]
\end{aligned}
$$

Action for the Wigner crystal

[R. Chitra, T. Giamarchi, P. Le Doussal Phys. Rev. B**65** 035312 (2001)]

$$S = \frac{1}{2\beta}\sum_{\omega_n}\int \frac{d^2\vec{q}}{(2\pi)^2} u_\alpha(\vec{q},\omega_n)\left[(\rho\omega_n^2 + \kappa q^2)\delta_{ab} + d\frac{q_\alpha q_\gamma}{q} + \frac{\rho_0 eB}{m}\omega_n\epsilon_{\alpha\gamma}\right] u_\gamma(-\vec{q},-\omega_n)$$

Rewrite using

$$\vec{u}(\vec{q}) = \hat{q}u_L(\vec{q}) + (\vec{z}\times\vec{q})u_T(\vec{q})$$

Gaussian Variational Method for the Wigner crystal

$$G_{cT}(q,i\omega_n)=\frac{cq^2+dq+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})}{\{[cq^2+dq+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})][cq^2+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})]+\rho_m^2\omega_n^2\omega_c^2\}}$$

$$G_{cL}(q,i\omega_n)=\frac{cq^2+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})}{\{[cq^2+dq+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})][cq^2+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})]+\rho_m^2\omega_n^2\omega_c^2\}},$$

$$G_{cLT}(q,i\omega_n)=\frac{\rho_m\omega_n\omega_c}{\{[cq^2+dq+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})][cq^2+\rho_m\omega_n^2+I(i\omega_n)+\Sigma(1-\delta_{n,0})]+\rho_m^2\omega_n^2\omega_c^2\}}, \quad (19)$$

where $I(i\omega_n)$ satisfies (in the semiclassical limit)

$$I(i\omega_n)=2\pi c\Sigma\int_{\mathbf{q}}\left[\frac{1}{cq^2+\Sigma}+\frac{1}{cq^2+dq+\Sigma}-\frac{2[cq^2+\omega_n^2+I(i\omega_n)+\Sigma]+dq}{[cq^2+\rho_m\omega_n^2+dq+I(i\omega_n)+\Sigma][cq^2+\rho_m\omega_n^2+I(i\omega_n)+\Sigma]+\rho_m^2\omega_n^2\omega_c^2}\right] \quad (20)$$

R. Chitra, T. Giamarchi, P. Le Doussal Phys. Rev. B**65** 035312 (2001)

Physical consequences

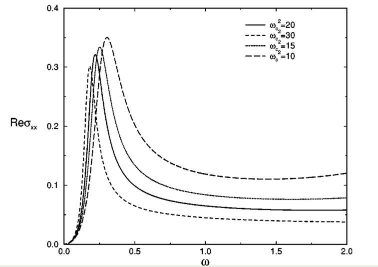
Quasi long range order

$$\overline{\langle e^{iG_0 u(\vec{x})} e^{-iG_0 u(\vec{0})} \rangle} \sim x^{-2}$$

Longitudinal onductivity

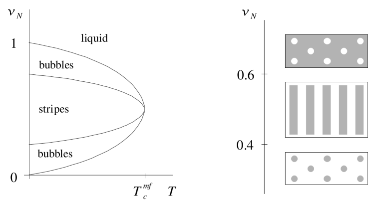
Cyclotron and pinning frequencies

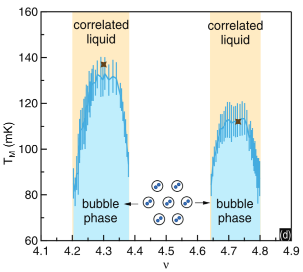
$\omega_{c,p} = (\sqrt{(eB/m)^2 + 4\Sigma/\rho_m} \pm eB/m)/2$



Other elastic quantum systems

Quantum Hall stripe and bubble phases





M. Fogler (2002) in *High Magnetic Fields: Applications in Condensed Matter Physics and Spectroscopy*

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Gaussian Variational Method for bubble and stripe phases

List of references

- M.-R. Li, H. A. Fertig, R. Côté, and Hangmo Yi Phys. Rev. Lett. **92**, 186804 (2004) [Stripes]
- Mei-Rong Li, H. A. Fertig, R. Côté, and Hangmo Yi Phys. Rev. B **71** 155312 (2005) [Stripes]
- R. Côté, Mei-Rong Li, A. Faribault, and H. A. Fertig Phys. Rev. B **72** 115344 (2005) [Bubbles]

Conclusion

Summary

- Quantum elastic phases with disorder
- Disorder pinning and a. c. conductivity
- Relation with classical elastic systems with columnar disorder
- Effect of Coulomb repulsion

Open problems

- Dislocations, melting
- Non-linear response (depinning transition, moving crystal)
- Out of equilibrium dynamics