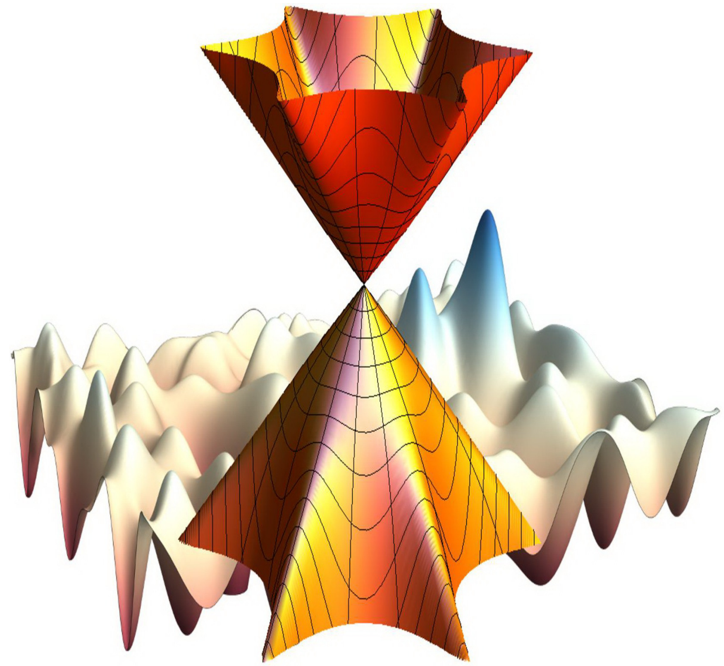
Meeting GDR "Interaction, Désordre, Elasticité" (Grenoble)

17-21 June 2024

Surface criticality in disorder-driven quantum transitions and beyond



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Outline

- Theory of surface critical phenomena
- Bragg glass in XY model and disordered periodic elastic systems
- Anderson localization
- Non-Anderson disorder-driven quantum transition in Dirac materials

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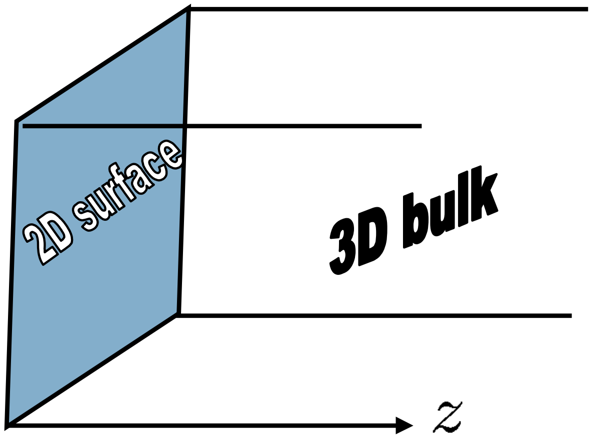
Theory of surface critical phenomena

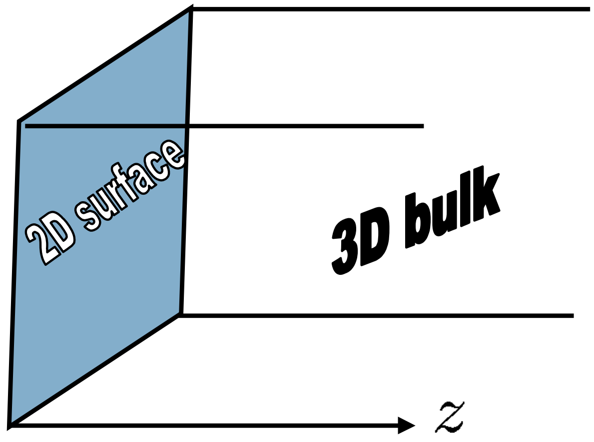


3D Ising model in a semi-infinite space

$$H = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \qquad J_{ij} = \begin{cases} J_s & \text{on the surface} \\ J_b & \text{in the bulk} \end{cases}$$

H.W. Diehl, in Phase transitions and critical phenomena, vol. 10, pp. 76-260 (1986)

Theory of surface critical phenomena



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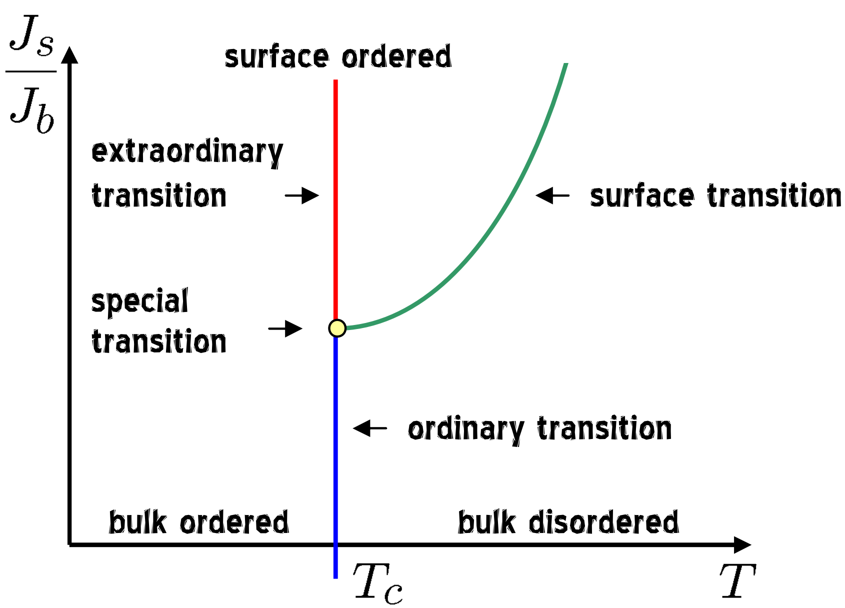
Phase diagram



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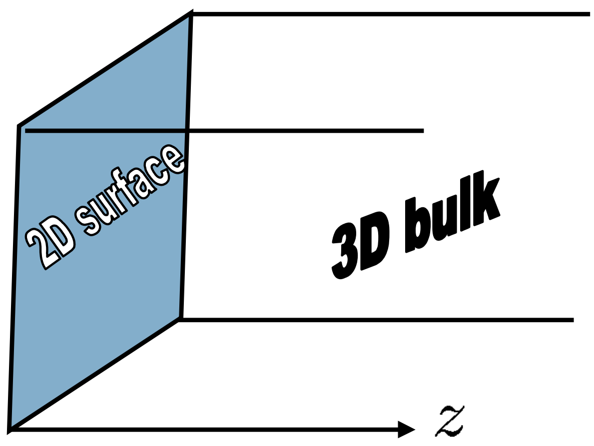

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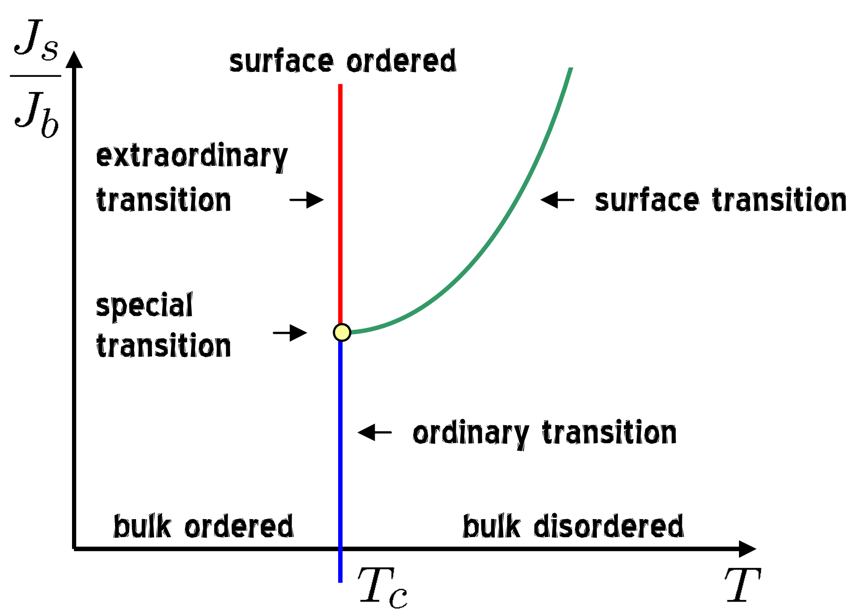

Critical exponents

order parameter

$$M_b \sim (T_c - T)^{\beta} \qquad \text{in the bulk}$$

$$M_s \sim (T_c - T)^{\beta_1} \qquad \text{on the surface}$$

correlation functions at T_c

$$\text{b-b} \sim \frac{1}{r^{d-2+\eta}}$$

$$\text{s-s} \sim \frac{1}{r^{d-2+\eta_{\parallel}}} \qquad \text{s-b} \sim \frac{1}{r^{d-2+\eta_{\perp}}}$$

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Mean field for the ordinary transition

Local magnetization in the mean field approximation

$$M(r) = \tanh\left(T^{-1}\sum_{r'} J(r,r')M(r')\right) \qquad J(r) = \sum_{r'} J(r,r') \qquad T_C = J(z\to\infty)$$

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Expanding in Taylor and assuming $J(z) = J(1 - \frac{1}{2\lambda}\delta(z))$ $\qquad \frac{1}{\lambda} \sim \frac{J_b - J_s}{J_b}$

$$\frac{1}{2}\frac{\partial^2 M(z)}{\partial z^2} = \tau M(z) + \frac{1}{3}M^3(z) \qquad \tau = \frac{(T-T_c)}{T_c}$$ (reduced temperature)

$$\frac{\partial M(0)}{\partial z} = \lambda^{-1}M(0)$$ (boundary conditions)

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Magnetization profile for $\tau < 0$

$$M(z) = (-\tau)^{1/2} f\left(\frac{z+\lambda}{\xi}\right)$$

Correlation length

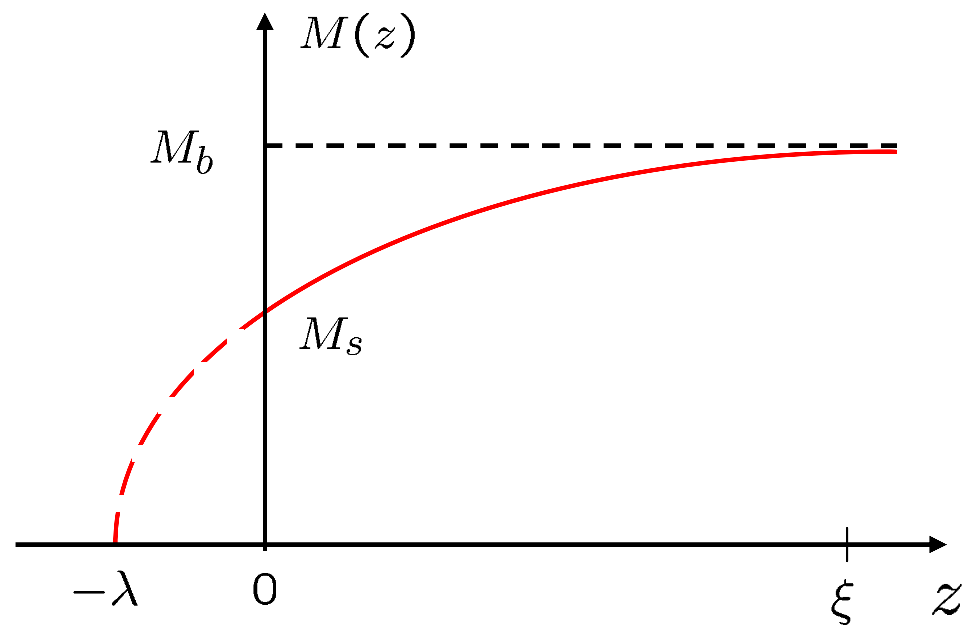
$$\xi = (-\tau)^{-1/2}$$

Bulk magnetization

$$M_b \sim (-\tau)^{1/2} \qquad \beta = 1/2$$

Surface magnetization

$$M_s \sim (-\tau) \qquad \beta_1 = 1$$



Correlation functions in the Gaussian approximation

Correlation function far in the bulk

$$\mathbf{r} = (\vec{x}, z)$$

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{[(\vec{x}_1 - \vec{x}_2)^2 + (z_1 - z_2)^2]^{(d-2)/2}}$$

$$\eta = 0$$

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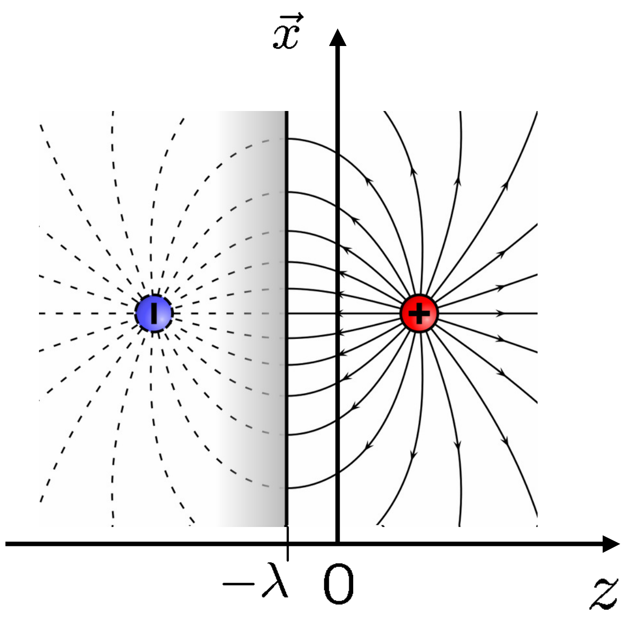
Correlation function close to the surface

Method of image charges



$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{[(\vec{x}_1 - \vec{x}_2)^2 + (z_1 - z_2)^2]^{(d-2)/2}} - \frac{1}{[(\vec{x}_1 - \vec{x}_2)^2 + (z_1 + z_2 + 2\lambda)^2]^{(d-2)/2}}$$

$$\text{b-b} \sim \frac{1}{r^{d-2}} \qquad \eta = 0$$

$$\text{s-s} \sim \frac{1}{x^d} \qquad \eta_{\parallel} = 2$$

$$\text{s-b} \sim \frac{1}{z^{d-1}} \qquad \eta_{\perp} = 1$$

$$\eta + \eta_{\parallel} = 2\eta_{\perp}$$

Renormalization group approach

ϕ^4 - **model**

$$H = \int d^{d-1}r \int_0^\infty dz \left[\frac{1}{2}(\nabla\phi)^2 + \frac{\tau_0}{2}\phi^2 + \frac{g_0}{4!}\phi^4\right]$$

Renormalization group approach

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Robin boundary conditions

$$\partial_z\phi|_{z=0} = c_0\,\phi|_{z=0}$$

Bare correlation function for $g = 0$

$$G_0(z,z';q) = \frac{1}{2\kappa_0}\left[e^{-\kappa_0|z-z'|} - \frac{c_0-\kappa_0}{c_0+\kappa_0}e^{-\kappa_0(z+z')}\right] \qquad \kappa_0 = \sqrt{q^2+\tau_0}$$

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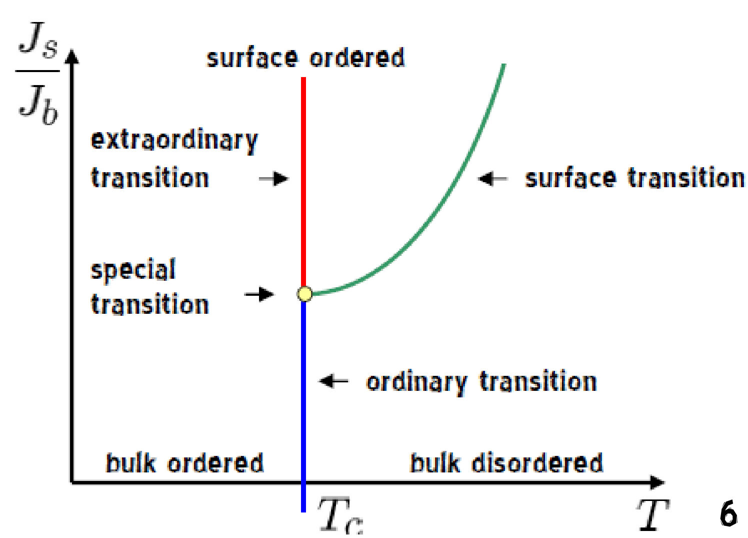
We have to renormalize not only $\tau_0,\ g_0,\ \phi$ but also $c_0,\ \phi|_{z=0}$

The renormalization group flow has 3 nontrivial fixed points $\tau^*,\ g^*$ with

$c^* = \infty$ Ordinary transition (Dirichlet boundary condition)

$c^* = 0$ Special transition (Neumann boundary condition)

$c \to -\infty$ Extraordinary transition (c is dangerously irrelevant)



Ordinary transition ($c^* = \infty$)

Expansion of the correlation function $G(z, z'; q) = \frac{1}{2\kappa_0}\left[e^{-\kappa_0|z-z'|} - e^{-\kappa_0(z+z')}\right] + \ldots$

UV singularities in correlation functions can be absorbed by renormalization

$$\phi = Z_\phi^{1/2}\phi_R \qquad \partial_z\phi|_s = (Z_\phi Z_1)^{1/2}\,\partial_z\phi|_{sR}$$

$$\tau_0 = \mu^2 Z_\tau \tau + \tau_c \qquad g_0 = \mu^{4-d} Z_g u$$

Renormalization conditions:

$$Z_\phi G^{(2,0)}(z, z'; q) = \text{finite}$$

$$Z_\phi Z_1^{1/2} \frac{\partial^2}{\partial z \partial z'} G^{(1,1)}(z, z'; q)\Big|_{z=0} = \text{finite}$$

$$Z_\phi = 1 - \frac{n+2}{36\varepsilon} g^2 + O(\varepsilon^3)$$

$$Z_1 = 1 + \frac{n+1}{3\varepsilon} g + O(\varepsilon^2)$$

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⇒

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RG functions and fixed point

$$\beta = \mu\partial_\mu g|_0$$

$$\eta_i = \mu\partial_\mu \ln Z_i|_0$$

⇒ $\beta(g^*) = 0$ ⇒

Critical exponents

$$\eta = \eta_\phi(g^*) \qquad \eta_\perp = (\eta + \eta_\parallel)/2$$

$$\eta_\parallel = 2 + \eta_1(g^*) \qquad \beta_1 = \nu(d - 2 + \eta_\parallel)/2$$

Surface critical exponents to one loop

$$\eta_\parallel = 2 - \frac{n+2}{n+8}\varepsilon \qquad \beta_1 = 1 - \frac{3}{2(n+8)}\varepsilon$$

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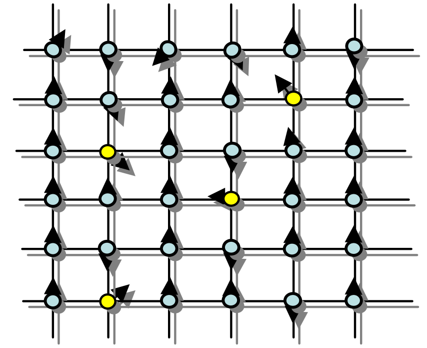
Semi-infinite Bragg glass

XY model with random fields in $d = 4 - \varepsilon$ dimension



$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \mathbf{S}_j - \sum_i \mathbf{h}_i \mathbf{S}_i - \mathbf{h}_1 \sum_{i \in \text{surface}} \mathbf{S}_i$$

$$|S_i|^2 = 1$$

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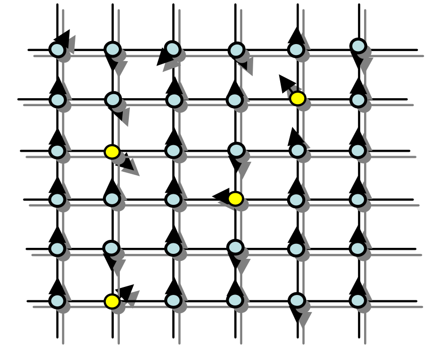

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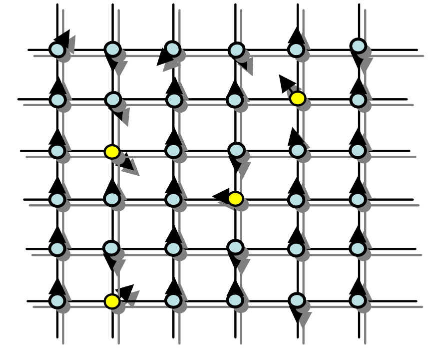
Replicated Hamiltonian averaged over disorder (continuum version)

D. S. Fisher, Phys. Rev. B 31, 7233 (1985)

$$\mathcal{H}_n = \int_V \left\{ \frac{1}{2} \sum_{a=1}^{n} (\nabla \mathbf{s}_a(\mathbf{r}))^2 - \frac{1}{2T} \sum_{a,b=1}^{n} \mathcal{R}\left(\mathbf{s}_a(r) \cdot \mathbf{s}_b(\mathbf{r})\right) \right\} - \sum_{a=1}^{n} \int_S \mathbf{h}_1 \cdot \mathbf{s}_a(\mathbf{x})$$

Semi-infinite Bragg glass



XY model with random fields in $d = 4-\varepsilon$ dimension

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Quasi-long range ordered phase for $d < 4$ can be studied by FRG

UV singularities in correlation functions can be absorbed by renormalization

Perturbative expansion

$$s = (\sqrt{1-\pi^2}, \pi)$$

$$\mathring{\pi} = Z_\pi^{1/2}\pi, \qquad \mathring{\pi}|_s = (Z_\pi Z_1)^{1/2}\, \pi|_s$$
$$\mathring{h} = \mu^2 Z_T Z_\pi^{-1/2} h, \qquad \mathring{h}_1 = \mu Z_T (Z_\pi Z_1)^{-1/2} h_1$$
$$\mathring{T} = \mu^{2-d} Z_T T, \qquad \mathring{R} = \mu^{4-d} K_d^{-1} Z_R[R]$$

RG functions

$$\beta[R] = -\,\mu \partial_\mu R(\phi)|_0$$

$$\zeta_i = \mu\partial_\mu \ln Z_i|_0\,, \quad (i = T, \pi, 1)$$

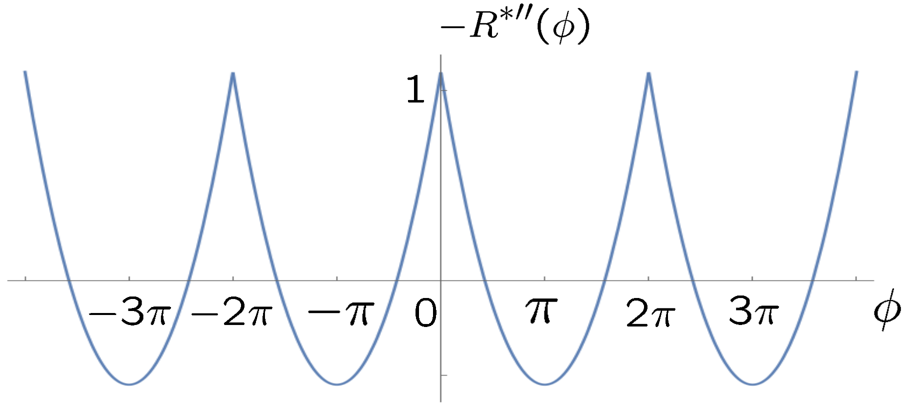
Fixed point

$$\partial_\ell R(\phi) \;=\; \varepsilon R(\phi) + \frac{1}{2}[R''(\phi)]^2 - R''(0)R''(\phi)$$

D. E. Feldman, PRB 61, 382 (2000)

P. Le Doussal, K.J. Wiese, PRL 96, 197202 (2006)

M. Tissier, G.Tarjus, PRB 74, 214419 (2006)



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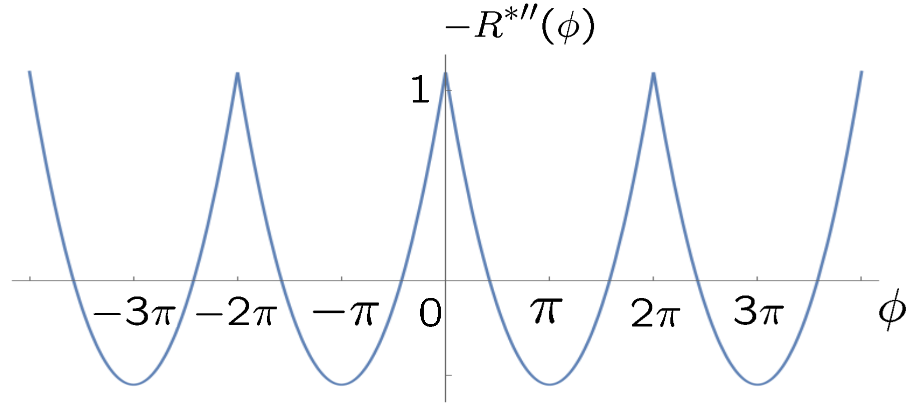

Connected two-point function

$$\overline{\langle \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}') \rangle - \langle \mathbf{s}(\mathbf{r}) \rangle \cdot \langle \mathbf{s}(\mathbf{r}') \rangle} \sim \frac{1}{|\mathbf{r}-\mathbf{r}'|^{d-2+\eta}}$$

Disconnected two-point function

$$\overline{\langle \mathbf{s}(\mathbf{r}) \rangle \cdot \langle \mathbf{s}(\mathbf{r}') \rangle} - \overline{\langle \mathbf{s}(\mathbf{r}) \rangle} \cdot \overline{\langle \mathbf{s}(\mathbf{r}') \rangle} \sim \frac{1}{|\mathbf{r}-\mathbf{r}'|^{d-4+\bar{\eta}}}$$

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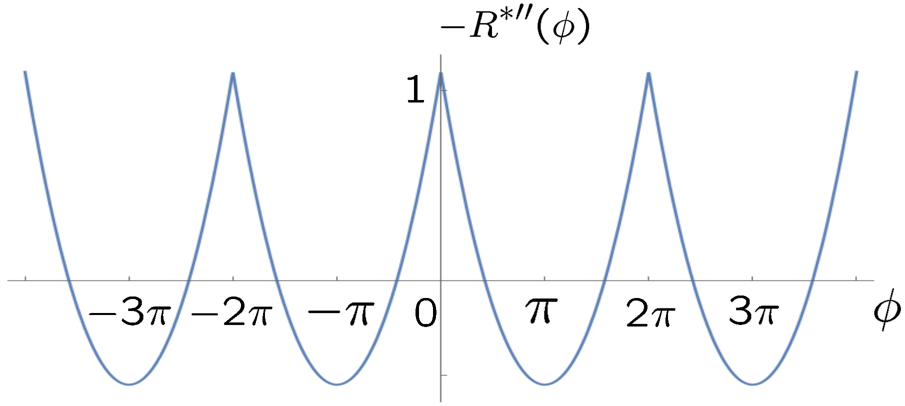

Connected two-point function

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$$\eta = \zeta_\pi^* - \zeta_T^*$$
$$\eta_\perp = \zeta_\pi^* + \zeta_1^*/2 - \zeta_T^*$$
$$\eta_\| = \zeta_\pi^* + \zeta_1^* - \zeta_T^*$$

Disconnected two-point function

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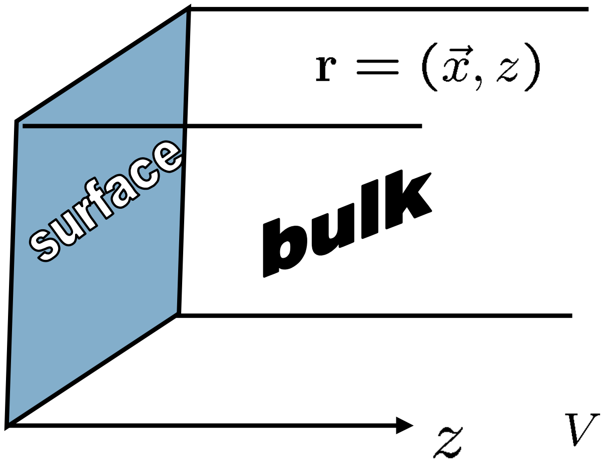
Critical exponents for the free surface $\mathrm{h}_1 \to 0$

AAF, Phys. Rev. E 86, 021131 (2012)

$$\eta = \frac{\pi^2}{9}\varepsilon \qquad \eta_\perp = \frac{\pi^2}{6}\varepsilon \qquad \eta_\| = \frac{2\pi^2}{9}\varepsilon$$

$$\bar\eta = \left(1+\frac{\pi^2}{9}\right)\varepsilon \qquad \bar\eta_\perp = \left(1+\frac{\pi^2}{6}\right)\varepsilon \qquad \bar\eta_\| = \left(1+\frac{2\pi^2}{9}\right)\varepsilon$$

Disordered periodic elastic systems



Hamiltonian

$$\mathcal{H} = \int d^{d-1}x \int_0^\infty z \left[\frac{c}{2}(\nabla u(\mathbf{r}))^2 + V(\mathbf{r}, u) \right]$$

c **elasticity constant**

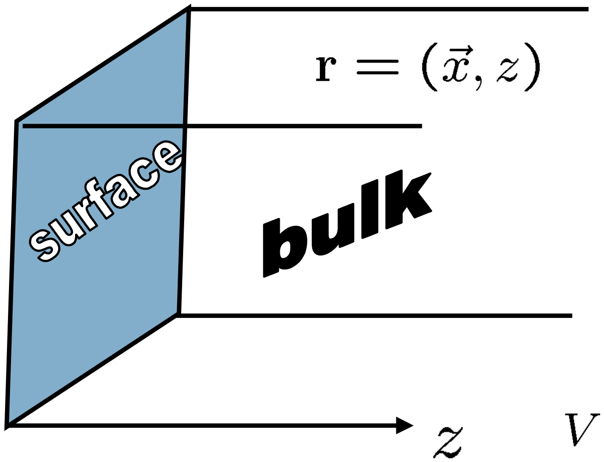
$V(x, u)$ **random potential with zero mean and variance**

$$\overline{V(x,u)V(x',u')} = R(u-u')\delta^d(x-x')$$

P. Le Doussal, K.J. Wiese,
P. Chauve, PRE 69, 026112 (2004)

K.J. Wiese, Rep. Prog. Phys. 85, 086502 (2022)

Disordered periodic elastic systems



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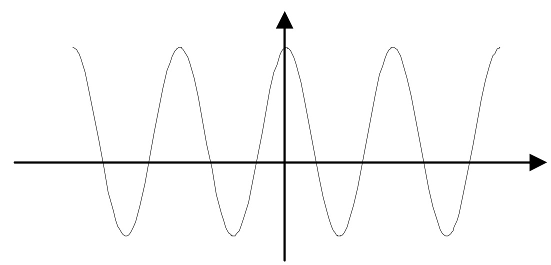
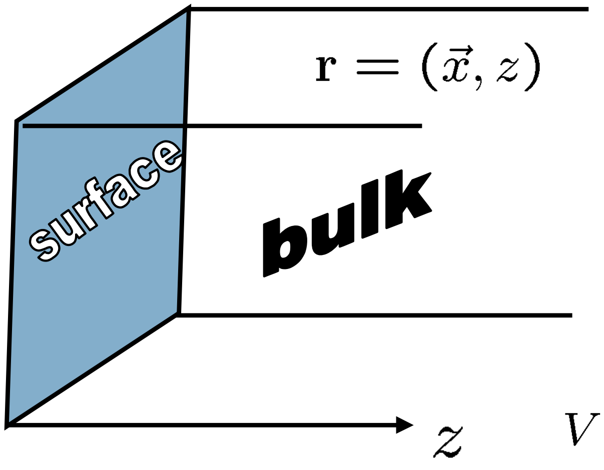
Random Periodic (RP): $R(u)$ **is periodic**

CDW, vortex lattice in type II superconductors



Disordered periodic elastic systems



Hamiltonian

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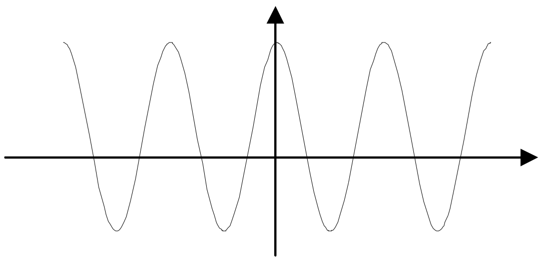

CDW, vortex lattice in type II superconductors

in the bulk
$$\overline{(u(\mathbf{r}) - u(\mathbf{r}'))^2} = \frac{\varepsilon}{18} \ln |\mathbf{r} - \mathbf{r}'|$$

close to the surface
$$\overline{(u(z) - u(0))^2} = \frac{\varepsilon}{12} \ln |z|$$

on the surface
$$\overline{(u(\vec{x}) - u(\vec{x}'))^2} = \frac{\varepsilon}{9} \ln |\vec{x} - \vec{x}'|$$

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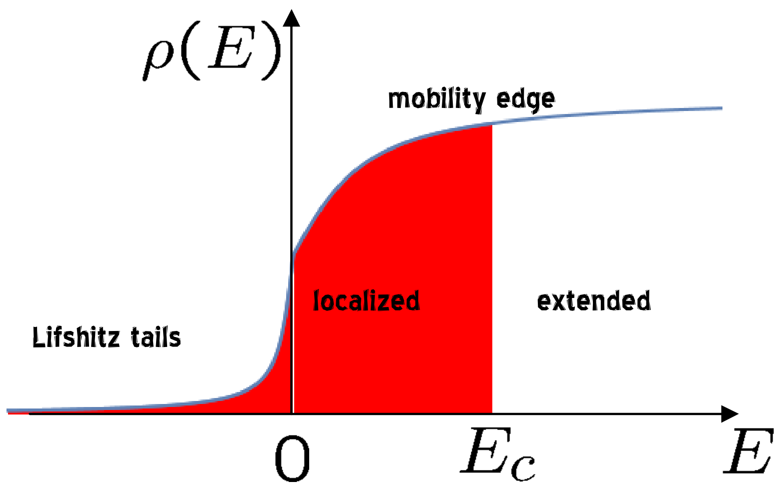
Anderson localization transition

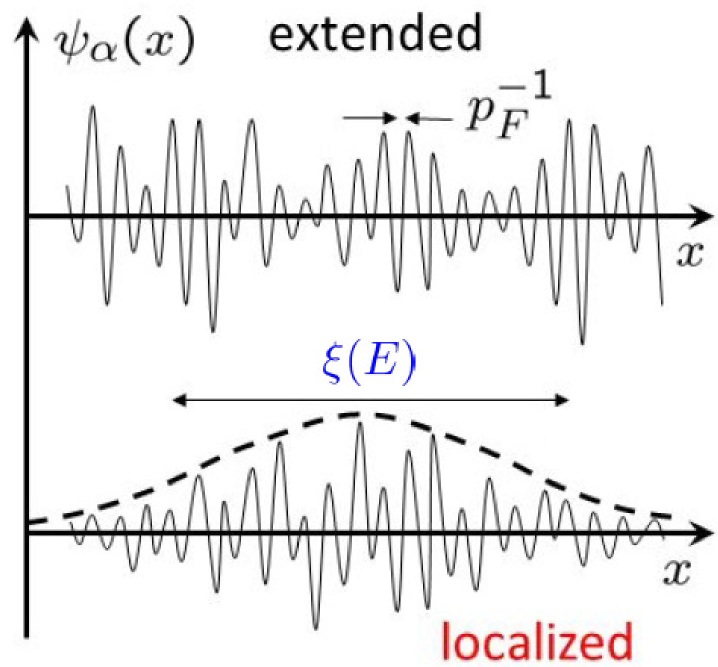
Single electron in a Gaussian random potential

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{x})\right]\psi_\alpha(\boldsymbol{x}) = E_\alpha\psi_\alpha(\boldsymbol{x})$$

$$\overline{V(\boldsymbol{x})} = 0$$

$$\overline{V(\boldsymbol{x})V(\boldsymbol{x}')} = \Delta\delta(\boldsymbol{x}-\boldsymbol{x}')$$





P.W. Anderson, Phys. Rev. 109, 1492 (1958)

Anderson localization transition

Single electron in a Gaussian random potential

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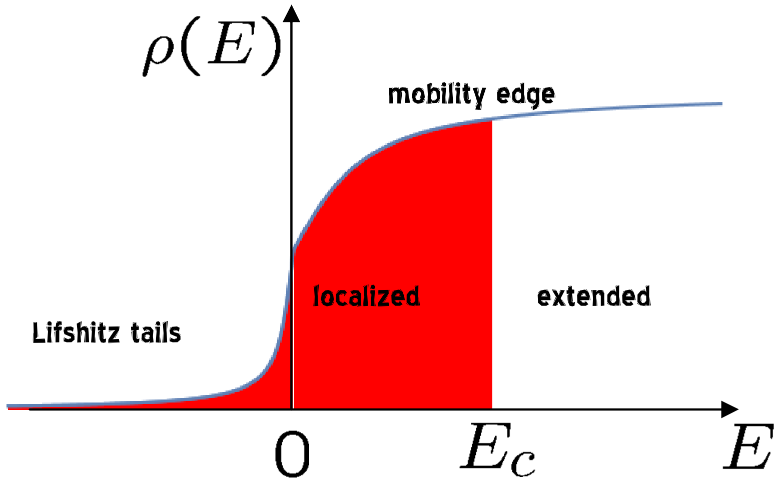

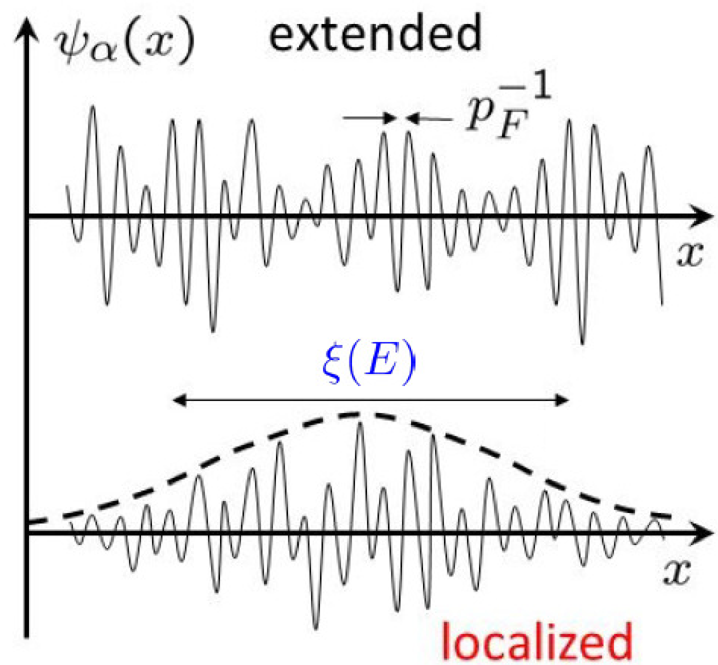

Localization length $\xi \sim (E_c - E)^{-\nu}$

Conductivity $\sigma \sim (E - E_c)^s \quad s = \nu(d-2)$

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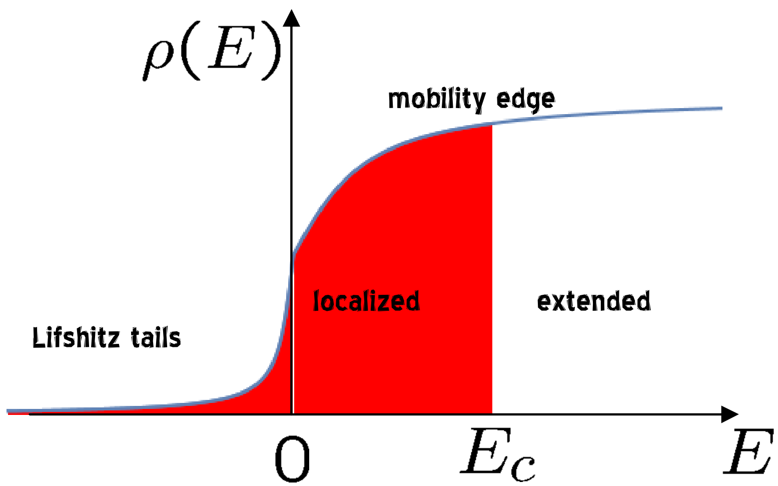

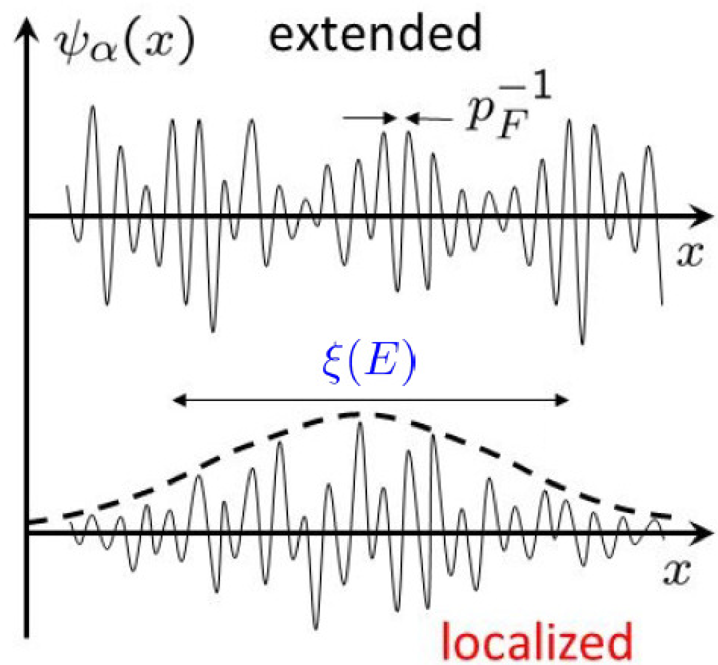

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P.W. Anderson, Phys. Rev. 109, 1492 (1958)

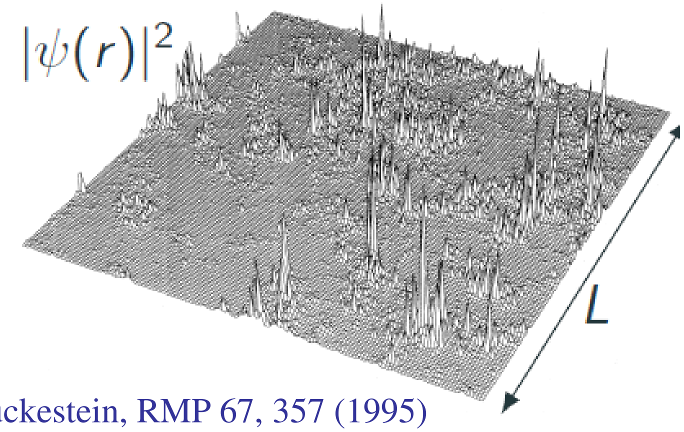
Field theory : Non-linear sigma model

$$S[Q] = \int d^dx \, \mathrm{Tr}\left[D(\nabla Q)^2 - 2i\Lambda Q\right] \qquad Q^2 = 1 \qquad \mathrm{Tr}Q = 0$$

F. Wegner, Z. Phys. B 35, 207 (1979)

Multifractality

Inverse participation ratio $P_q = \int \mathrm{d}^d r \overline{|\psi(\boldsymbol{r})|^{2q}}$



B. Huckestein, RMP 67, 357 (1995)

$$P_q \sim \begin{cases} L^{-d(q-1)} & \text{(extended states)} \\ L^{-d(q-1)-\tilde{\Delta}_q} & \text{(critical wave functions)} \\ L^0 & \text{(localized states)} \end{cases}$$

Multifractal spectrum

$$\tilde{\Delta}_q^{(\mathrm{O})} = q(1-q)\varepsilon + \mathcal{O}(\varepsilon^4)$$

F. Evers, A. D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008)

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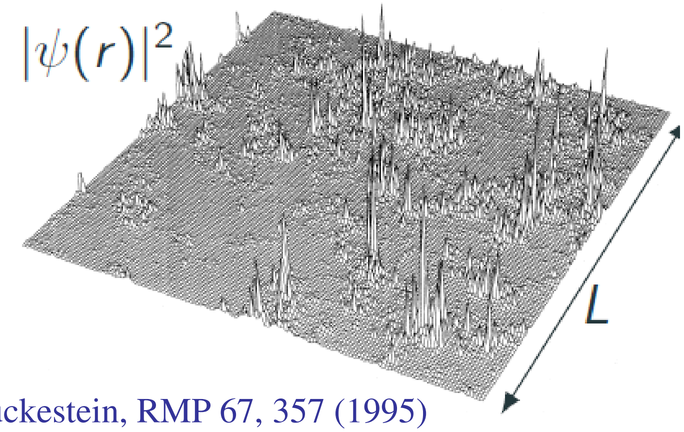

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Multifractal spectrum in the bulk $\tilde{\Delta}_q^{(\mathrm{O})} = q(1-q)\varepsilon + \mathcal{O}(\varepsilon^4)$

F. Evers, A. D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008)

What about the surface?

Surface multifractal spectrum $P_q^{(s)} = \int \mathrm{d}^{d-1} x \overline{|\psi(\boldsymbol{x})|^{2q}} \sim L^{-d(q-1)+1-\tilde{\Delta}_q^{(s)}}$

2D weakly localized metallic system with dimensionless conductance $g \gg 1$ shows multifractality on length scales below the localization length $\xi \sim e^{\pi g}$

Bulk multifractal spectrum $\tilde{\Delta}_q = (\pi g)^{-1} q(1-q)$

Surface multifractal spectrum $\tilde{\Delta}_q^{(s)} = 2(\pi g)^{-1} q(1-q)$

A. R. Subramaniam, I. A. Gruzberg, A. W. W. Ludwig, F. Evers, A. Mildenberger, A. D. Mirlin, PRL 96, 126802 (2006)

Outline

- Theory of surface critical phenomena
- Bragg glass in XY model and disordered periodic elastic systems
- Anderson localization
- Non-Anderson disorder-driven quantum transition in Dirac materials

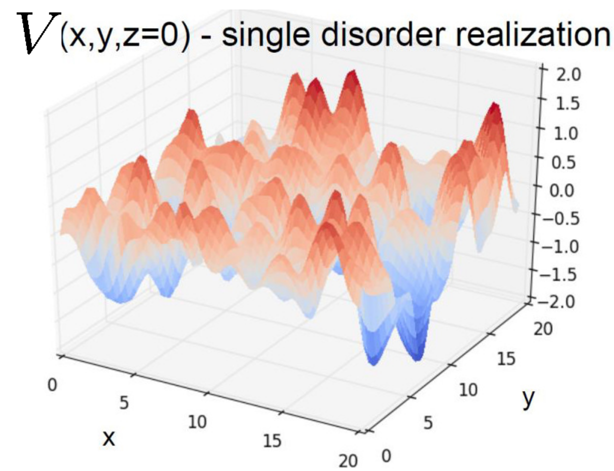
Disordered Dirac fermions

Hamiltonian

$$\hat{H} = -iv_F\vec{\alpha}\vec{\partial} + V(x)$$

Gaussain random potential :

$$\overline{V(x)} = 0 \qquad \overline{V(x)V(x')} = \Delta\delta(x - x')$$



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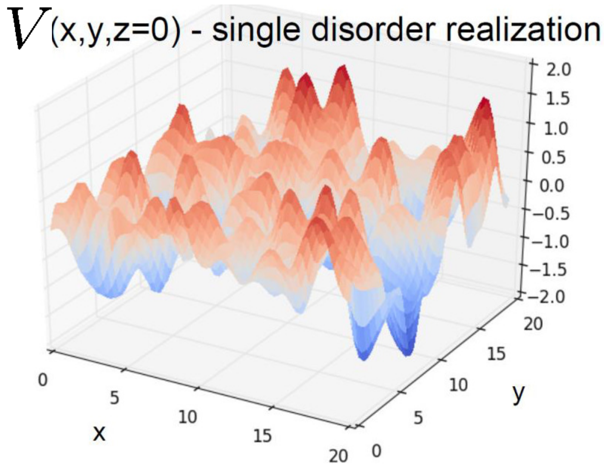
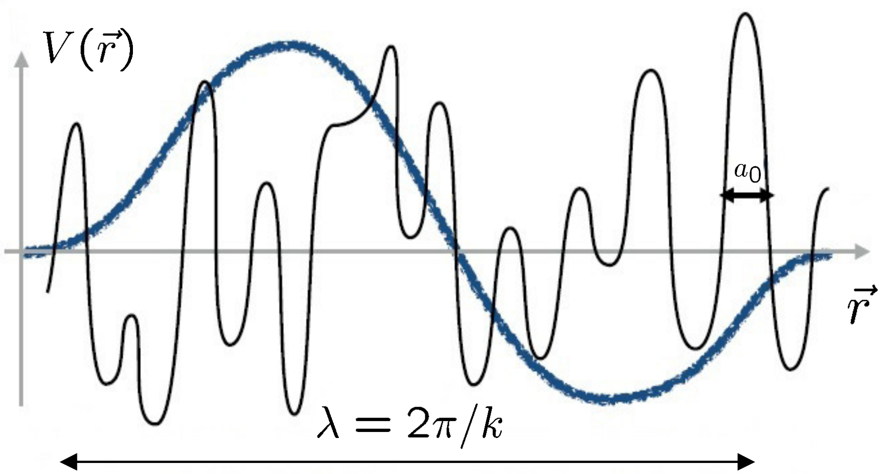


Scaling arguments



Kinetic energy : $E_{typ} = \hbar v_F k$

Disorder potential : $V_{typ} \sim \sqrt{\Delta}\left(\frac{\lambda}{a_0}\right)^{-d/2}$

In the limit of zero energy ($k \to 0$)
disorder is dominant for $d < 2$
and irrelevant for $d > 2$

Disordered Dirac fermions

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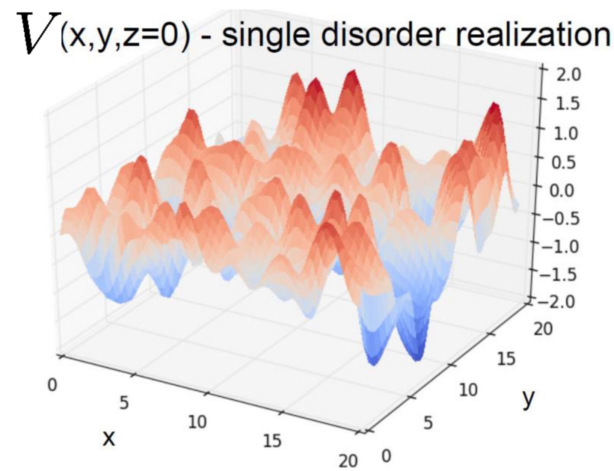


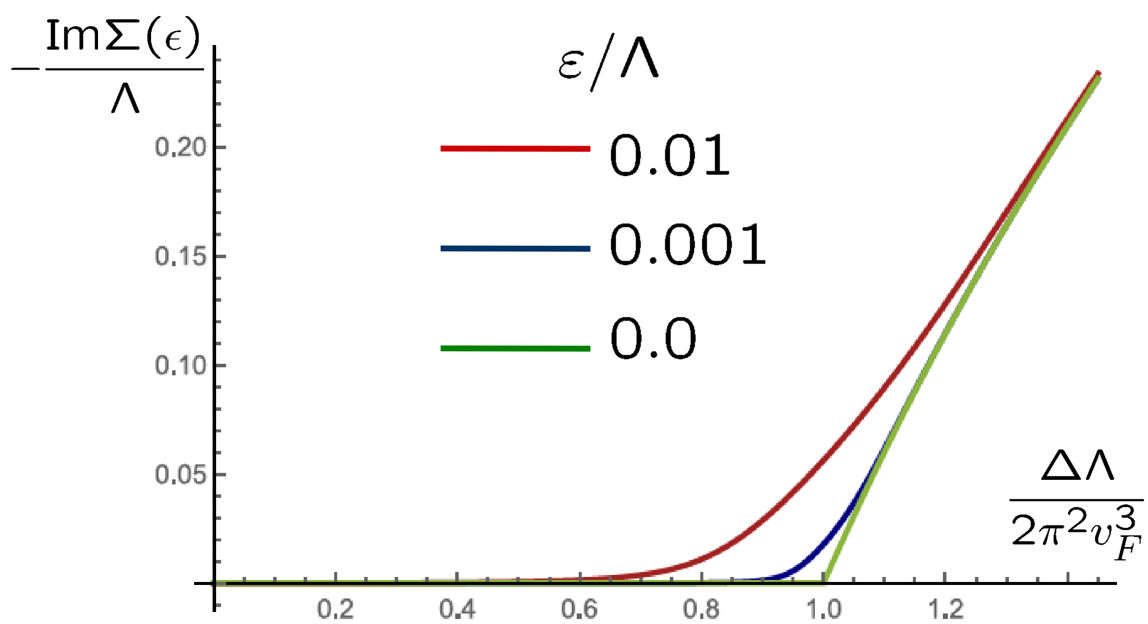
Self-consistent Born approximation

Green function

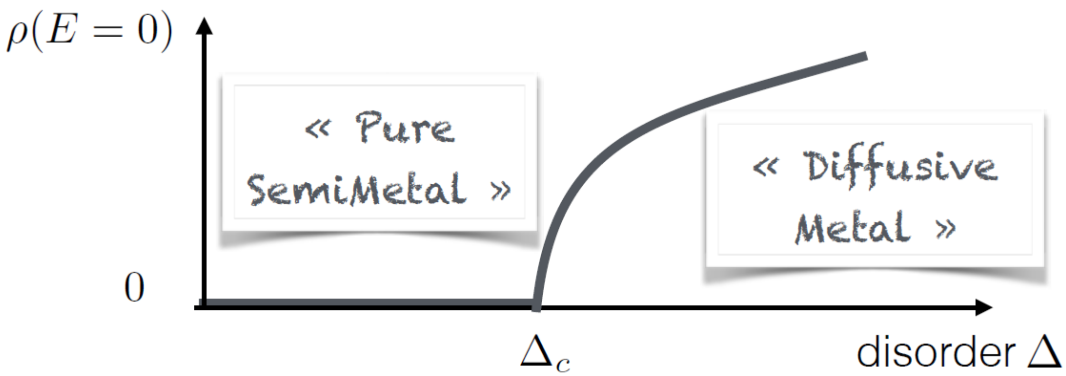
$$G(\vec{k}, \epsilon) = \frac{1}{\epsilon - v_F\vec{\alpha}\vec{k} - \Sigma(\vec{k}, \epsilon)}$$

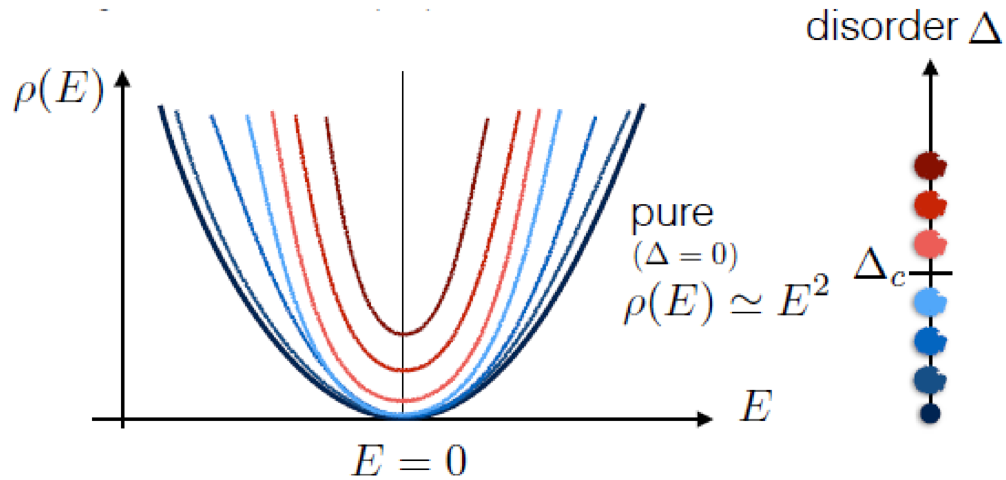
SCBA equation

$$\Sigma(\epsilon) = \Delta \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\left[G(\vec{k}, \epsilon)\right]$$



New disorder driven quantum transition ≠ Anderson localization

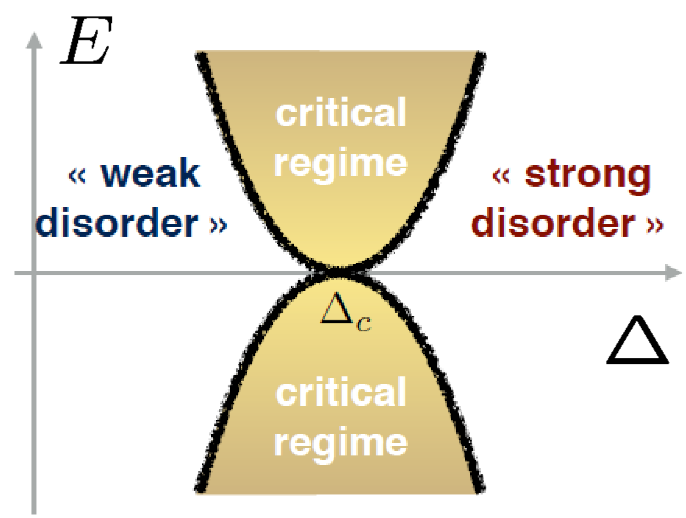




The mean free path $\xi \sim |\Delta - \Delta_c|^{-\nu}$

Density of states (DOS) :

$$\rho(E) = \xi^{z-d}\tilde{\rho}(E\xi^z, (\Delta - \Delta_c)\xi^{1/\nu})$$



at the transition :

$$\rho(0, \Delta) \sim |\Delta - \Delta_c|^{\beta} \qquad \beta = \nu(d - z)$$

$$\rho(E) \sim E^{d/z-1}$$

K. Kobayashi, T. Ohtsuki, K.-I. Imura, I. F. Herbut, PRL 112, 016402 (2014)

Replicated action : Gross-Neveu model in the limit of $N \to 0$

$$S_{\mathrm{GN}} = \int d^d x \left[-i \sum_{a=1}^{N} \bar{\psi}_a \alpha_j \partial_j \psi_a - \frac{1}{2} \Delta \sum_{a,b=1}^{N} \left(\bar{\psi}_a \psi_a \right) \left(\bar{\psi}_b \psi_b \right) \right]$$

Replicated action : **Gross-Neveu model in the limit of** $N \to 0$

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Renormalization group in $d = 2 + \varepsilon$

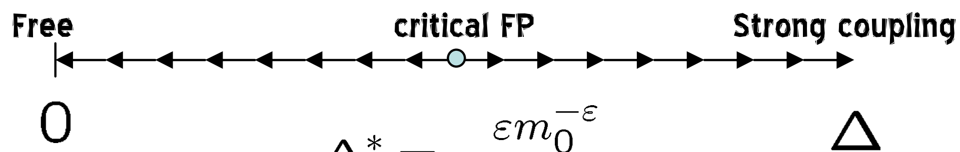
B. Roy, S. Das Sarma, PRB 90, 241112(R) (2014)

$$-m \partial_m \tilde{\Delta} = \beta(\tilde{\Delta}) = -\varepsilon \tilde{\Delta} + 4(1 - \tilde{N}) \tilde{\Delta}^2 + \ldots$$

$$\Delta = \tilde{\Delta} m^{-\varepsilon} \qquad \tilde{N} = \frac{N}{2} \mathrm{tr} \mathbb{I}$$

$$\frac{1}{\nu} = \beta'(\Delta^*)$$

RG flow



$$\Delta^* = \frac{\varepsilon m_0^{-\varepsilon}}{4(1 - \tilde{N})}$$

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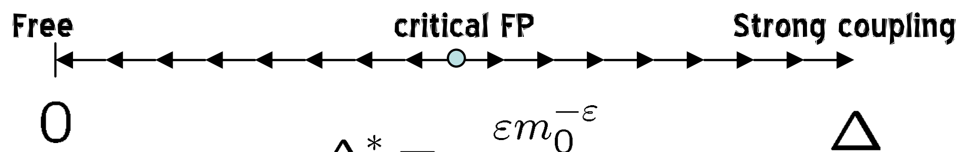


$$\Delta^* = \frac{\varepsilon m_0^{-\varepsilon}}{4(1-\tilde{N})}$$

Critical exponents

$$\frac{1}{\nu} = \varepsilon + \frac{\varepsilon^2}{2} + \frac{3\varepsilon^3}{8} + O\left(\varepsilon^4\right)$$

$$z = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \frac{3\varepsilon^3}{32} + O(\varepsilon^4)$$

$$\eta = -\frac{\varepsilon^2}{8} + \frac{3\varepsilon^3}{16} - \frac{25\varepsilon^4}{128} + O(\varepsilon^5)$$

Multifractal spectrum

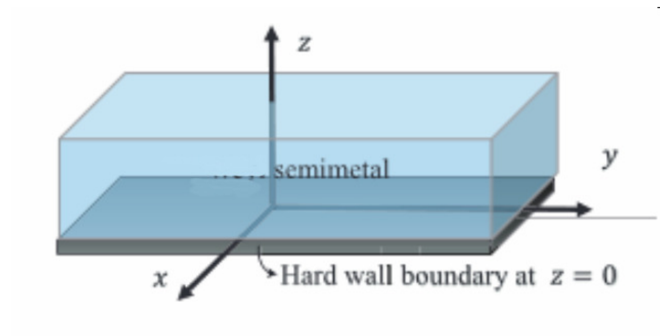
$$\tilde{\Delta}_q^{\mathrm{Dirac}} = \frac{3}{8} q(1-q)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

T. Louvet, AAF, D. Carpentier, PRB 94, 220201(R) (2016)

S.V. Syzranov, V. Gurarie, L. Radzihovsky, Ann. Phys. 373, 694 (2016)

E. Brillaux, D. Carpentier, AAF, PRB 100, 134204 (2019)

Dirac fermions in a semi-infinite system



E. Witten, Three Lectures On Topological Phases Of Matter, 2018

Hamiltonian

$$\hat{H}_0 = i\tau_z \boldsymbol{\sigma}\cdot\boldsymbol{\partial} \qquad z > 0 \qquad \alpha_i = \tau_z \sigma_i$$

Boundary conditions

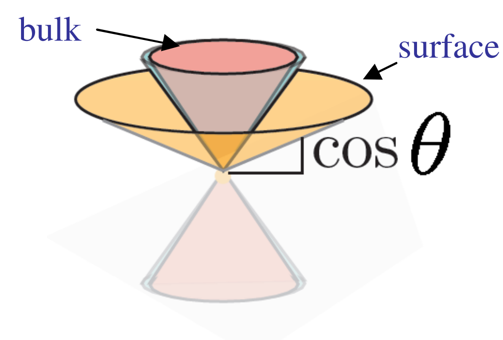
$$M\psi|_{z=0+} = \psi|_{z=0+}$$

Unitary Hermitian, no transverse current

$$\{M, \tau_z \sigma_z\} = 0$$

$$M_\theta = \begin{pmatrix} 0 & 0 & ie^{i\theta} & 0 \\ 0 & 0 & 0 & -ie^{-i\theta} \\ -ie^{-i\theta} & 0 & 0 & 0 \\ 0 & ie^{i\theta} & 0 & 0 \end{pmatrix}$$

Surface states



$$\hat{H}_0 \psi = \epsilon \psi \qquad \psi \sim e^{-\mu z}$$

$$\varepsilon = k_{||} \cos\theta$$

$$\mu = k_{||} \sin\theta$$

O. Shtanko, L. Levitov, PNAS 115, 5908 (2018)

Disordered Dirac fermions in a semi-infinite system

Hamiltonian & boundary conditions

$$\hat{H} = -i\tau_z \boldsymbol{\sigma} \cdot \boldsymbol{\partial} + V(\boldsymbol{x})$$

$$M_\theta \psi|_{z=0} = \psi|_{z=0}$$

$$\overline{V(\boldsymbol{x})} = 0$$

$$\overline{V(\boldsymbol{x})V(\boldsymbol{x}')} = \Delta\delta(\boldsymbol{x} - \boldsymbol{x}')$$

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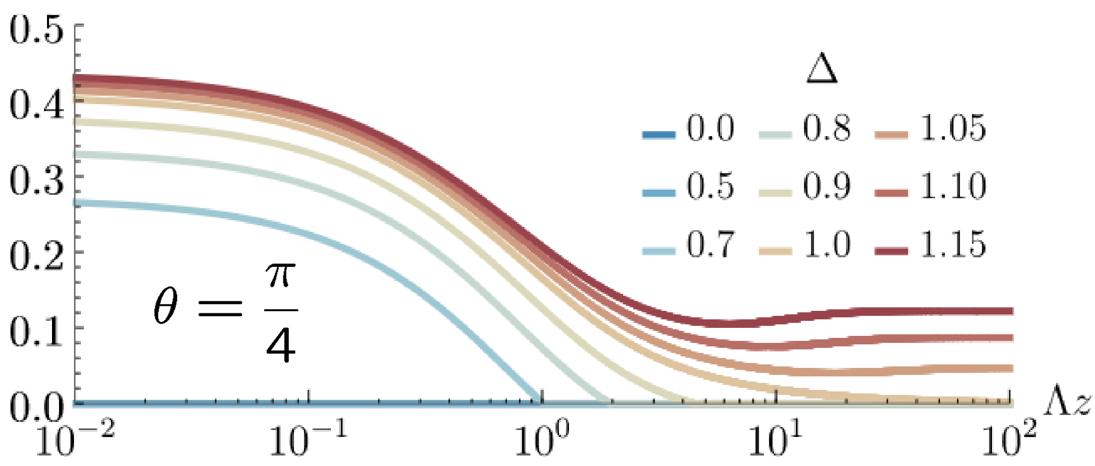
$$M_\theta \psi|_{z=0} = \psi|_{z=0} \qquad \overline{V(\boldsymbol{x})V(\boldsymbol{x}')} = \Delta\delta(\boldsymbol{x}-\boldsymbol{x}')$$

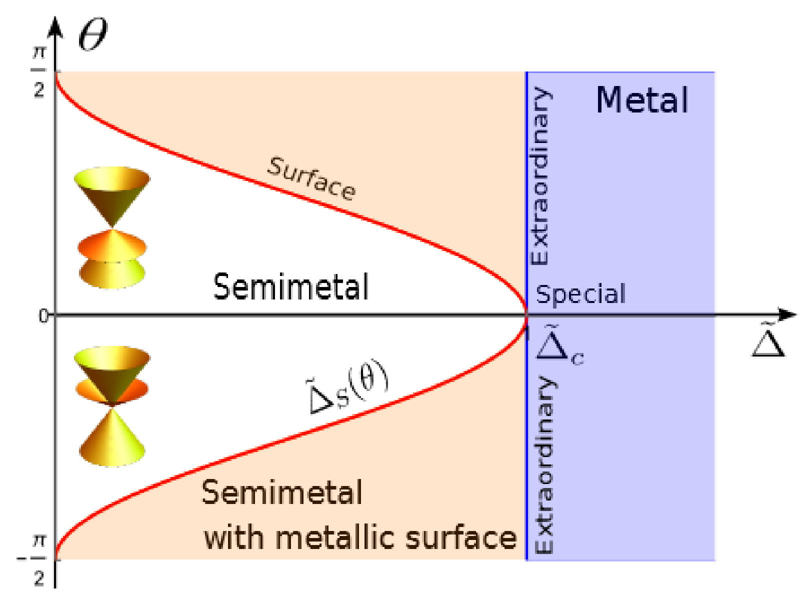
Local self-consistent Born approximation

$$\Sigma(\epsilon, z) = \Delta \int \frac{d^2k}{(2\pi)^2} \mathrm{Tr}\left[G(\vec{k}, z, z, \epsilon)\right]$$

Local DOS profile $\rho(\epsilon = 0, z)$



Phase diagram in the presence of bulk disorder



E. Brillaux, AAF, PRB 100, 103, 081405 (2021)

Special transtion $(\theta = 0)$: renormalization group

Action for the system with a surface

$$S = -i\int_{z>0} d^d x\,\bar{\psi}_a(\boldsymbol{x})\alpha_\mu\partial_\mu\psi_a(\boldsymbol{x}) - \frac{\Delta}{2}\int_{z>0} d^d x\,\bar{\psi}_a(\boldsymbol{x})\psi_a(\boldsymbol{x})\bar{\psi}_b(\boldsymbol{x})\psi_b(\boldsymbol{x})$$
$$+i\int d^{d-1}r\bar{\psi}_a(\vec{r})\alpha_z M\psi_a(\vec{r})$$

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$$+i\int d^{d-1}r\bar\psi_a(\vec r)\alpha_z M\psi_a(\vec r)$$

Renormalization

$$\mathring{\psi} = Z_\psi^{1/2}\psi,\quad \mathring{\psi}_s = Z_{\psi_s}^{1/2}\psi_s,\quad \mathring{O} = Z_\omega Z_\psi^{-1}O,\quad \mathring{O}_s = Z_{O_s}Z_{\psi_s}^{-1}O_s,\quad \mathring{\Delta} = \frac{2\mu^{-\varepsilon}}{K_d}\frac{Z_\Delta}{Z_\psi^2}\Delta$$

$$O(x) := \bar\psi(x)\psi(x),\quad O_s(r) := \bar\psi_s(r)\psi_s(r)$$

Z-factors from minimal subtraction scheme

$$Z_\psi = 1 - \frac{\Delta^2}{\varepsilon}$$

$$Z_\omega = 1 + \frac{2\Delta}{\varepsilon} + \frac{6\Delta^2}{\varepsilon^2}$$

$$Z_\Delta = 1 + \frac{4\Delta}{\varepsilon} + \Delta^2\left(\frac{16}{\varepsilon^2} + \frac{2}{\varepsilon}\right)$$

$$Z_{\psi_s} = 1 - \frac{2\Delta}{\varepsilon} + O(\Delta^2)$$

$$Z_{O_s} = 1 - \frac{6\Delta}{\varepsilon} + O(\Delta^2)$$

E. Brillaux, AAF, I. Gruzberg, PRB 109, 174204 (2024)

Special transtion: renormalization group

RG functions

$$\beta(\Delta) = -\mu \frac{\partial \Delta}{\partial \mu}\bigg|_{\mathring{\Delta}}, \ \eta_i(\Delta) = -\beta(\Delta)\frac{\partial \ln Z_i}{\partial \Delta}, \quad (i = \psi, \psi_s, \omega, O_s), \ \gamma(\Delta) = \eta_\omega(\Delta) - \eta_\psi(\Delta)$$

Critical exponents at fixed point $\beta(\Delta^*) = 0$

$$\frac{1}{\nu} = \beta'(\Delta^*), \ z = 1 + \gamma(\Delta^*), \ \eta = \eta_\psi(\Delta^*), \ \eta_\parallel = \eta_{\psi_s}(\Delta^*)$$

$$\beta = \nu(d - z), \quad \beta_s = \nu\Big(d - 1 - \eta_{O_s}(\Delta^*) + \eta_{\psi_s}(\Delta^*)\Big)$$

Two point surface functions

$$G(r_1, r_2) = \frac{i}{S_d}(1 + M_0)\frac{\vec{\alpha}\cdot(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^{d+\eta_\parallel}}$$

$$G(0, z) = -\frac{i}{S_d}(1 + M_0)\frac{\alpha_z}{z^{d-1+\eta_\perp}}$$

$$\eta = -\frac{\varepsilon^2}{8} + O(\varepsilon^3)$$

$$\eta_\parallel = -\frac{\varepsilon}{2} + O(\varepsilon^2)$$

$$\eta_\perp = \frac{1}{2}(\eta + \eta_\parallel)$$

Surface DOS

$$\rho_s \sim |\Delta - \Delta_c|^{\beta_s}$$

$$\frac{\beta_s}{\beta} = 1 + \frac{3}{2}\varepsilon + O(\varepsilon^2)$$

E. Brillaux, AAF, I. Gruzberg, PRB 109, 174204 (2024)

Thank you for your attention!